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WEIERSTRASS' THEOREM AND KNESER'S THEOREM ON TRANSVERSALS FOR THE MOST GENERAL CASE OF AN EXTREMUM OF A SIMPLE DEFINITE INTEGRAL[*]

BY

OSKAR BOLZA

Introduction.

Let

$$y = Y(x, a) \tag{1}$$

donote a one-parameter set of extremals for the integral

$$J = \int_{x_0}^{x_1} F(x, y, y')\,dx,$$

passing *through a fixed point* (ξ, η) and furnishing a field about an arc $\mathfrak{E}_0$ of the particular extremal $a = a_0$, and let

$$a = \mathfrak{a}(x, y)$$

denote the inverse function of the field, obtained by solving (1) with respect to a.

Further, let the function $W(x, y)$ be defined by the two equations

$$U(x, a) = \int_{\xi}^{x} f(x, Y, Y')\,dx,$$

$$W(x, y) = U(x, \mathfrak{a}),$$

so that the function $W(x, y)$—which we will call the *field integral*—represents the value of the integral J taken from the fixed point (ξ, η) to an arbitrary point (x, y) of the field along the unique extremal of the field passing through the point (x, y).

Then the following two fundamental formulas hold for the partial derivatives of the field integral :[†]

$$\begin{aligned} \frac{\partial W}{\partial x} &= f(x, y, p) - p f_{y'}(x, y, p), \\ \frac{\partial W}{\partial y} &= f_{y'}(x, y, p), \end{aligned} \tag{2}$$

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[*] Presented to the Society at the New Haven summer meeting September 4, 1906. Received for publication May 11, 1906.

[†] See, for instance, BOLZA, *Lectures on the Calculus of Variations*, p. 266.

where $p = p(x, y)$ is the slope of the extremal of the field passing through (x, y), at that point:

$$p(x, y) = Y'(x, \alpha).$$

We shall call these formulas *Hamilton's Formulas*, because they have first been given by HAMILTON* as early as 1835, even for a much more general case.

From HAMILTON's formulas follow immediately: on the one hand "*Weierstrass' theorem*" †

$$\Delta J = \int_{x_0}^{x_1} E dx,$$

by means of "WEIERSTRASS' construction"; on the other hand "*Beltrami-Hilbert's independence theorem*," ‡ which states that the expression

$$(3) \qquad [f(x, y, p) - p f_{y'}(x, y, p)]\, dx + f_{y'}(x, y, p) dy$$

is a complete differential if the slope-function $p(x, y)$ is substituted for p.

When one attempts to extend HAMILTON's formulas and the allied theory from the particular case of a set of extremals through a fixed point to the case of *any set of extremals*, one meets at once with a difficulty. It is, in this case, not a priori clear from what point as lower limit the integral W should be taken on the different extremals of the set. This difficulty has been removed by KNESER as follows: § He first takes the integral $W(x, y)$ from an arbitrary but fixed curve $\mathfrak{T}$ drawn across the field and computes the partial derivatives of W; it turns out that they differ from the simple expressions (2) only by an additional term in the expression of $\partial W / \partial y$. And then he so determines the curve $\mathfrak{T}$ that this additional term vanishes. He finds as solution of this problem — which we shall call "*Kneser's Problem*" — the result that the curve $\mathfrak{T}$ must be a "*transversal*" to the set of extremals forming the field. Closely connected with this result is "*Kneser's Theorem on Transversals.*" ||

HAMILTON's formulas being thus extended to any set of extremals, the corresponding extension of WEIERSTRASS' theorem on the one hand, and of BELTRAMI-HILBERT's independence theorem on the other hand follow easily.

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* Philosophical Transactions, 1835, part I, p. 77.

† WEIERSTRASS, *Berlin Lectures*, 1879; see, for instance, OSGOOD, Annals of Mathematics, ser. 2, vol. 2 (1901), p. 115, and BOLZA, *Lectures*, pp. 87, 266.

‡ The theorem was first given, independently of HAMILTON's formulas, by BELTRAMI in a paper on geodesics, *Sulla teoria delle linee geodetiche*, published in 1868 (Rendiconti del Reale Istituto Lombardo, ser. 2, vol. 1, pp. 708–718, also *Opere*, vol. 1, pp. 366–373). This important paper seems to have remained unnoticed until quite recently (my own attention has been called to it by Professor KNESER). The theorem was rediscovered thirty years later by HILBERT who made it the basis of his well-known proof of WEIERSTRASS' theorem (see Göttinger Nachrichten, 1900, pp. 253–297, and Archiv für Mathematik und Physik, ser. 3, vol. 1 (1901), p. 231.)

§ See KNESER, *Lehrbuch der Variationsrechnung*, § 15, and OSGOOD, l. c., p. 119.

|| See KNESER, *Lehrbuch*, p. 48, and BOLZA, *Lectures*, § 33.

A second method for the extension of WEIERSTRASS' theorem to any set of extremals has been given by HILBERT.* Instead of starting from the integral W, he starts from the differential expression (3) with an arbitrary function p of x and y, and proposes so to determine the function p that the expression (3) becomes a complete differential. He finds as solution of this problem — which we shall call (according to A. MAYER) "*Hilbert's Problem*" — the slope-function

$$p = Y'(x, a),$$

derived from *any* set of extremals, and bases on this result a new proof of WEIERSTRASS' theorem by means of his *Invariant Integral.*

HILBERT's theory has recently been extended by A. MAYER † to the so-called most general case of an extremum of a simple definite integral, in which it is required to minimize an integral of the form

$$(4) \qquad J = \int_{x_0}^{x_1} f(x, y_1, \cdots, y_n; y_1', \cdots, y_n')\, dx,$$

involving n unknown functions $y_1, \cdots, y_n$ of x and their first derivatives $y_1', \cdots, y_n'$, connected by $r < n$ differential equations

$$(5) \qquad f_\rho(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0 \qquad (\rho = 1, 2, \cdots, r)$$

MAYER finds the remarkable result that for $n > 1$ not all, but only a certain class of n-parametric sets of extremals, furnish solutions of HILBERT's problem, and derives from his results a proof of WEIERSTRASS' theorem by means of HILBERT's invariant integral.

In the present paper, which has developed out of the study of Professor MAYER's memoirs, *I propose to give an analogous extension, to the same general case, of Weierstrass' original theory and of Kneser's theory* as sketched above.

I have endeavored to reach in the discussion of the general case the same degree of rigor which has been attained for the simplest type of problems. For this purpose, I make use of certain existence-theorems concerning implicit functions and systems of differential equations, which extend the results, usually given for the vicinity of *a point*, to the vicinity of *a point set;* these auxiliary theorems are given in §§ 1 and 2.

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*See the references to HILBERT's papers on p. 460, third footnote; compare also OSGOOD, loc. cit., p. 121, and BOLZA, *Lectures*, § 21.

† *Ueber den Hilbert'schen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale*, Leipziger Berichte, 1903, pp. 131-145; 1905, pp. 49-67 and 1905, pp. 313-314. We shall refer to these papers as "MAYER, I, II, III."

HILBERT himself has given a geometrical proof of the generalized independence theorem for the case $r = 0$, in which he reduces the case n to the case $n - 1$; see his paper, *Zur Variationsrechnung*, Göttinger Nachrichten, February, 1905.

In § 3 follows a short sketch of the reduction of EULER-LAGRANGE's differential equations to a canonical system, in §§ 4 and 5 the proof of WEIERSTRASS' theorem for a set of extremals through a fixed point, based upon a combination of HAMILTON's formulas and WEIERSTRASS' construction.

In §§ 6 and 7 the extension of KNESER's theory is given, and finally § 8 contains the application to WEIERSTRASS' theorem for MAYER's class of n-parametric sets of extremals.

§ 1. *Auxiliary theorems concerning implicit functions.*

For brevity I shall say that a function $f(x_1, \cdots, x_n)$, which is defined in* a region† $\mathbf{A}$, is of class $C^{(p)}$ in $\mathbf{A}$, when the function f itself and its partial derivatives up to the pth order (inclusive) exist and are continuous in* $\mathbf{A}$ (if necessary, after a proper extension‡ of the definition of the function $f(x_1, \cdots, x_n)$ beyond the region $\mathbf{A}$). Further I shall say that a point $(x) = (x_1, \cdots, x_n)$ lies in "*the vicinity* (ρ) *of a point set* $\mathbf{M}$" defined in the $(x_1, \cdots, x_n)$ space, if it lies in the vicinity (ρ) of at least one point of $\mathbf{M}$, i. e., if there exists a point $(\bar{x}_1, \cdots, \bar{x}_n)$ of $\mathbf{M}$ such that

$$|x_1 - \bar{x}_1| < \rho, \cdots, |x_n - \bar{x}_n| < \rho.$$

The vicinity (ρ) of the set $\mathbf{M}$ thus defined will be denoted by $(\rho)_{\mathbf{M}}$.

The following theorem holds:

LEMMA I. *Let the functions* $f_i(x_1, \cdots, x_n)$ *be of class* C' *in a region* $\mathbf{A}$; *let further* $\mathbf{C}$ *be a bounded*§ *closed*|| *point set in the interior*¶ *of* $\mathbf{A}$ *which has the property that any two distinct points* (x'), (x'') *of* $\mathbf{C}$ *are transformed into two distinct points* (u'), (u'') *by the transformation*

$$u_i = f_i(x_1, \cdots, x_n) \qquad (i = 1, 2, \cdots, n). \tag{6}$$

If then the Jacobian

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

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* *in* signifies *at all points of.*

† I use the word "region" for a point set which contains "inner" points, in accordance with BLISS, Annals of Mathematics, ser. 2, vol. 6 (1905), p. 49.

‡ This additional clause is necessary if we wish to cover the case where the region $\mathbf{A}$ contains also "boundary points," because the derivative of a function, as usually defined, has a meaning only for inner points of the region in which the function is defined.

§ "Borné" (JORDAN).

|| "Abgeschlossen" (CANTOR).

¶ I. e., every point of $\mathbf{C}$ is an "inner" point of $\mathbf{A}$.

is different from zero in[*] **C**, *a vicinity* $(\rho)_{\mathbf{C}}$ *contained in the interior of* **A** *can be assigned such that*

1) *the transformation* (6) *defines a one-to-one correspondence between the point set* $(\rho)_{\mathbf{C}}$ *and its image* $\mathbf{S}_\rho$ *in the* (u)*-space;*

2) *the set* $\mathbf{S}_\rho$ *is made up exclusively of inner points;*

3) *the inverse functions*

$$x_k = \psi_k(u_1, \cdots, u_n)$$

thus defined for $\mathbf{S}_\rho$ *are of class* C' *in* $\mathbf{S}_\rho$.

The proof of the theorem is based upon a slight modification of the method by which I have proved a special case of the theorem in § 34 of my *Lectures on the Calculus of Variations;* for the details I refer to my note, *Ein Satz über eindeutige Abbildung und seine Anwendung in der Variationsrechnung*, in one of the forthcoming numbers of the Mathematische Annalen.

If we denote by **E** the image of **C** in the (u)-space, it follows that we can assign a second positive quantity σ such that $(\sigma)_{\mathbf{E}}$ is contained in $\mathbf{S}_\rho$. Hence we have the

COROLLARY: *For every point* (u) *in* $(\sigma)_{\mathbf{E}}$ *the equations* (6) *have one and but one solution* (x) *in* $(\rho)_{\mathbf{C}}$.

This remark leads to the following

LEMMA II.[†] *If the functions* $f_i(x_1, \cdots, x_m;\ y_1, \cdots, y_n)$ *are of class* C' *in a region* **A** *in the* (x, y)*-space and if the n equations*

$$f_i(x_1, \cdots, x_n;\ y_1, \cdots, y_n) = 0 \qquad (i = 1, 2, \cdots, n), \tag{7}$$

are fulfilled for all the points of a point set **C** *of the following properties:*

1) *the set* **C** *is bounded and closed, and lies in the interior of* **A**;

2) *if* (x', y'), *and* (x'', y'') *are two distinct points of* **C**, *then always* $(x') \neq (x'')$;

3) *the Jacobian*

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

is different from zero in **C**;

then, if **X** *denote the "projection"*[‡] *of* **C** *into the* (x)*-space, two positive quantities* ρ (*first*) *and* σ (*second*) *can be assigned such that*

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[*] I. e., at every point of **C**.

[†] This is an extension of the ordinary theorem on implicit functions which refers to the solution of a system of equations in *the vicinity of a point*, see GENOCCHI–PEANO, *Differentialrechnung und Grundzüge der Integralrechnung*, nos. 110–117; JORDAN, *Cours d'Analyse*, vol. 1, nos. 91, 92; OSGOOD, *Lehrbuch der Functionentheorie*, vol. 1, pp. 47–55.

[‡] The projection of the point $(x'_1, \cdots, x'_m, y'_1, \cdots, y'_n)$ into the (x)-space is the point $(x'_1, \cdots, x'_m)$.

1) *for every* (x) *in* $(\sigma)_X$ *there exists one and but one solution* (y) *of the equations* (7) *for which* (x, y) *lies in* $(\rho)_C$;

2) *the corresponding implicit functions*

$$y_i = \phi_i(x_1, \cdots, x_n)$$

are of class C' *in* $(\sigma)_X$.

To prove lemma II apply lemma I to the system of equations

$$u_h = x_h, \qquad u_{m+i} = f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) \quad (h = 1, 2, \cdots, m; i = 1, 2, \cdots, n),$$

and put afterwards $u_{m+i} = 0$.

§ 2. *An existence theorem for systems of ordinary differential equations.*

We consider a system of n differential equations

$$(8) \qquad \frac{dx_k}{dt} = f_k(t, x_1, \cdots, x_n) \qquad (k = 1, 2, \cdots, n),$$

in which the functions f_k are continuous in a region **A** and admit continuous first partial derivatives with respect to $x_1, \cdots, x_n$ in the interior of **A**.

Then if $A(\tau, \xi_1, \cdots, \xi_n)$ is a point in the interior of the region **A**, there exists, according to CAUCHY's existence theorem, a unique solution of (8) which passes through the point A and which is of class C' in the vicinity of $t = \tau$. We denote* it by

$$x_k = \phi_k(t; \tau, \xi_1, \cdots, \xi_n),$$

so that

$$\phi_k(\tau; \tau, \xi_1, \cdots, \xi_n) = \xi_k.$$

Under these circumstances the following theorem holds:

LEMMA III. *Let*

$$(9) \qquad x_k = x_k^0(t), \qquad t_0 \leqq t \leqq t_1$$

be a solution of (8) *which lies* IN THE INTERIOR OF THE REGION **A**; *let* τ_0 *be a particular value of* t *in the interval* $(t_0 t_1)$ *and put*

$$\xi_k^0 = x_k^0(\tau_0).$$

Then three positive quantities d, e_0, e_1 *can be assigned such that for every point* $A(\tau, \xi_1, \cdots, \xi_n)$ *for which*

$$|\tau - \tau_0| \leqq d, \qquad |\xi_k - \xi_k^0| \leqq d \qquad (k = 1, 2, \cdots, n),$$

the solution

$$(10) \qquad x_k = \phi_k(t; \tau, \xi_1, \cdots, \xi_n)$$

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*I adopt the notation of BLISS in his article, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, ser. 2, vol. 6 (1905), pp. 49–68. To the same article the reader is referred for the literature of the subject.

exists, is of class C' and lies in the interior of **A** FOR EVERY t IN THE INTERVAL

$$t_0 - e_0 \leqq t \leqq t_1 + e_1 .$$

For $\tau = \tau_0$, $\xi_1 = \xi_1^0$, $\cdots$, $\xi_n = \xi_n^0$, *the solution* (10) *coincides with the solution* (9):

$$\phi_k(t;\, \tau_0,\, \xi_1^0,\, \cdots,\, \xi_k^0) \equiv x_k^0(t), \qquad \text{in } (t_0 t_1).$$

For analytic functions f_k the theorem has been given, in a slightly different form, by KNESER in his *Lehrbuch der Variationsrechnung* (§ 27), for non-analytic functions by A. C. LUNN in his Thesis (Chicago, 1904), as yet unpublished. On account of the importance of the theorem for the Calculus of Variations, I add here still another proof:

Since the solution (10) is supposed to lie in the interior of the region **A**, it can be "continued"* beyond the interval $(t_0 t_1)$ to a slightly larger interval $(t_0 - e_0,\, t_1 + e_1)$ without leaving the interior of **A**, and without ceasing to be of class C'. From theorems on continuous functions it follows then that we can assign a positive quantity σ such that the domain $[\sigma]$ defined by the inequalities

$$[\sigma]: \qquad t_0 - e_0 \leqq t \leqq t_1 + e_1, \qquad |x_k - x_k^0(t)| \leqq \sigma \quad (k = 1, 2, \cdots, n),$$

still lies in the interior of **A**. In this domain $[\sigma]$ the functions f_k are continuous and satisfy LIPSCHITZ's condition †

$$(11) \quad |f_k(t, x_1', \cdots, x_n') - f_k(t, x_1'', \cdots, x_n'')| < K \sum_{i=1}^{n} \left| x_i' - x_i'' \right| \quad (k = 1, 2, \cdots, n),$$

with a certain value of the constant K.

We now coördinate with every point $(t, x_1, \cdots, x_n)$ of the domain

$$\tilde{\mathbf{A}}: \qquad t_0 - e_0 \leqq t \leqq t_1 + e_1, \qquad -\infty < x_k < +\infty \quad (k = 1, 2, \cdots, n),$$

a point $(t, \tilde{x}_1, \cdots, \tilde{x}_n)$ of $[\sigma]$ by means of the following definition:

$$\tilde{x}_i = \begin{cases} x_i & \text{when} \quad x_i^0(t) - \sigma \leqq x_i \leqq x_i^0(t) + \sigma, \\ x_i^0(t) + \sigma & \text{when} \quad x_i > x_i^0(t) + \sigma, \\ x_i^0(t) - \sigma & \text{when} \quad x_i < x_i^0(t) - \sigma, \end{cases}$$

and define ‡ the functions $\tilde{f}_k(t, x_1, \cdots, x_n)$ for the domain $\tilde{\mathbf{A}}$ by the equation

$$\tilde{f}_k(t, x_1, \cdots, x_n) = f_k(t, \tilde{x}_1, \cdots, \tilde{x}_n).$$

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* Compare BLISS, loc. cit., p. 52.

† Compare BLISS, loc. cit., p. 50, footnote, and p. 60, footnote.

‡ The definition of $\tilde{f}_k$ admits of a simple geometrical interpretation in the case $n = 1$, if the value of $\tilde{f}_k$ is represented by a third coördinate perpendicular to the (t, x)-plane.

From this definition it follows easily that the functions $\tilde{f}_k$ are continuous in the domain $\tilde{\mathbf{A}}$; moreover there exists a positive quantity G such that

$$|\tilde{f}_k(t, x_1, \cdots, x_n)| \leqq G \qquad \text{in } \tilde{\mathbf{A}};$$

and finally the functions $\tilde{f}_k$ satisfy LIPSCHITZ's condition in $\tilde{\mathbf{A}}$ with the same value of the constant K which occurs in (11).

Hence follows, according to CAUCHY's existence theorem: If

$$t_0 - e_0 \leqq \tau \leqq t_1 + e_1$$

and if $\xi_1, \cdots, \xi_n$ is *any* system of values, the system of differential equation,

$$\frac{dx_k}{dt} = \tilde{f}_k(t, x_1, \cdots, x_n) \qquad (k=1, 2, \cdots, n), \tag{12}$$

has a unique solution passing through the point $A(\tau, \xi_1, \cdots, \xi_n)$:

$$x_k = \tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n),$$

and this solution exists and is of class C' *throughout the whole* interval* $(t_0 - e_0, t_1 + e_1)$.

Moreover it follows from PEANO's inequality† that

$$\begin{aligned} &|\tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n) - \tilde{\phi}_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0)| \\ &\qquad < e^{nK|t-\tau|} \left\{ G|\tau - \tau_0| + \sum_{i=1}^{n} |\xi_i - \xi_i^0| \right\}. \end{aligned} \tag{13}$$

On the other hand it follows from the definition of the functions $\tilde{f}_k$ that every solution of (12), which lies wholly in $[\sigma]$, satisfies at the same time (8), and vice versâ. Hence we infer in the first place that the solution

$$x_k = \phi_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0)$$

of (8) — which coincides with the solution (9) since both pass through the point $(\tau_0, \xi_1^0, \cdots, \xi_n^0)$ — satisfies also (12); therefore we have

$$\tilde{\phi}_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0) \equiv \phi_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0) \equiv x_k^0(t).$$

It follows now from (13) that if we take the positive quantity d sufficiently small and choose τ and ξ_k so that

$$|\tau - \tau_0| \leqq d, \qquad |\xi_k - \xi_k^0| \leqq d \qquad (k=1, 2, \cdots, n), \tag{14}$$

the solution

$$x_k = \tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n) \qquad (t_0 - e_0 \leqq t \leqq t_1 + e_1),$$

of (12) lies in $[\sigma]$ and satisfies therefore at the same time (8), and since it satisfies the same initial conditions as (10), the two solutions must be identical; hence also the solution (10) exists, is of class C' and lies in the interior of **A** for every t in the interval $(t_0 - e_0, t_1 + e_1)$, if the condition (14) is satisfied, Q.E.D.

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* Compare PICARD, *Traité d'Analyse*, vol. 3 (ed. 1896), p. 91.

† Compare BLISS, loc. cit., pp. 55 and 62.

From the general existence theorems* for differential equations it follows further that the functions

$$\phi_k(t;\ \tau,\ \xi_1,\ \cdots,\ \xi_n)$$

considered as functions of their $n + 2$ arguments are continuous together with the derivatives

$$\frac{\partial\phi_k}{\partial t},\quad \frac{\partial\phi_k}{\partial\tau},\quad \frac{\partial\phi_k}{\partial\xi_i},\quad \frac{\partial^2\phi_k}{\partial t\,\partial\tau},\quad \frac{\partial^2\phi_k}{\partial t\,\partial\xi_i}$$

throughout the domain

$$t_0 - e_0 \leqq t \leqq t_1 + e_1,\qquad |\tau - \tau_0| \leqq d,\qquad |\xi_k - \xi_k^0| \leqq d\qquad (k=1, 2, \cdots, n),$$

and that the Jacobian

$$\frac{\partial(\phi_1, \phi_2, \cdots, \phi_n)}{\partial(\xi_1, \xi_2, \cdots, \xi_n)}$$

is different from zero throughout the same domain.

§ 3. *The differential equations of the problem and their general integral.*

After these preliminaries we turn to the problem stated in the introduction.

We suppose the functions $f, f_1, \cdots, f_r$ to be of class C'' in a certain region **T** of the $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-space. According to EULER-LAGRANGE's rule† we set

$$\Omega = f + \sum_{\rho=1}^{r} \lambda_\rho f_\rho,$$

where the λ's are unknown functions of x, and write for brevity

$$(15)\qquad \Omega_i = \frac{\partial\Omega}{\partial y_i},\qquad \Omega_{n+i} = \frac{\partial\Omega}{\partial y_i'},\qquad \Omega_{2n+\rho} = \frac{\partial\Omega}{\partial\lambda_\rho} \equiv f_\rho.$$

We start in our developments from the assumption that we have found a solution

$$(16)\quad y_i = y_i(x),\quad \lambda_\rho = \lambda_\rho(x),\quad x_0 \leqq x \leqq x_1\qquad (i=1, 2, \cdots, n;\ \rho=1, 2, \cdots, r)$$

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[*] Compare BLISS, loc. cit., pp. 61, 63, 67.

[†] Compare A. MAYER, Mathematische Annalen, vol. 26 (1886), p. 74, and Leipziger Berichte, 1895, p. 129; TURKSMA, Mathematische Annalen, vol. 47 (1896), p. 33; KNESER, *Lehrbuch der Variationsrechnung*, § 56–58; HAHN, Monatshefte für Mathematik und Physik, vol. 14 (1903), p. 326; HILBERT, Göttinger Nachrichten (1905), p. 1; compare also KNESER's article in the *Encyclopædie*, IIA, p. 579, 580.

of EULER-LAGRANGE's differential equations, i. e., of the system of $n+r$ differential equations

$$\text{(I)} \qquad \Omega_i - \frac{d}{dx}\Omega_{n+i} = 0, \qquad f_\rho = 0 \quad (i=1,2,\cdots,n;\ \rho=1,2,\cdots,r),$$

of the following properties:

A) the functions $y_i(x)$ are *of class* C'', the functions $\lambda_\rho(x)$ *of class* C' in the interval $(x_0 x_1)$.

B) the curve

$$y_i = y_i(x), \qquad y_i' = y_i'(x), \qquad x_0 \leqq x \leqq x_1 \quad (i=1,2,\cdots,n),$$

in the $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-space lies *in the interior of the region* **T**.

C) the *Jacobian*

$$D(x, y_1, \cdots, y_n;\ y_1', \cdots, y_n';\ \lambda_1, \cdots, \lambda_r) = \frac{\partial(\Omega_{n+1}, \cdots, \Omega_{2n}, f_1, \cdots, f_r)}{\partial(y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r)}$$

is *different from zero* along the solution (16), i. e.,

$$D[x, y_1(x), \cdots;\ y_1'(x), \cdots;\ \lambda_1(x), \cdots] \neq 0$$

throughout the interval $(x_0 x_1)$.

With the system (I) is associated a "canonical system"* of differential equations which is obtained as follows:

We apply lemma II to the problem of solving the $n+r$ equations†

$$\text{(17)} \quad \Omega_{n+i}(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots) = v_i;\ \ f_\rho(x, y_1, \cdots;\ y_1', \cdots) = 0$$

with respect to

$$y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r.$$

To the region **A** of the lemma corresponds in the present case the region

$$(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \text{ in } \mathbf{T};\ -\infty < \lambda_\rho < +\infty;\ -\infty < v_i < +\infty,$$

and to the point set **C** the curve

$$\text{(18)} \qquad y_i = y_i(x),\ y_i' = y_i'(x),\ \lambda_\rho = \lambda_\rho(x),\ v_i = v_i(x),\ x_0 \leqq x \leqq x_1,$$

where $v_i(x)$ is defined by

$$\text{(19)} \qquad v_i(x) = \Omega_{n+i}[x, y_1(x), \cdots;\ y_1'(x), \cdots;\ \lambda_1(x), \cdots].$$

On account of our assumptions *A*), *B*), *C*) the curve (18) satisfies all the conditions required by the lemma. Hence there exist two positive quantities ϵ, δ, such that for every $x, y, \cdots, y_n, v_1, \cdots, v_n$ in the vicinity (δ) of the curve

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* Compare KNESER's article in the *Encyclopædie*, IIA, pp. 584-586 and the references there given; also JORDAN, *Cours d'Analyse*, vol. 3, no. 375, and A. MAYER's papers referred to in the introduction and his paper in vol. 69 of the Journal für Mathematik, p. 240; we adopt, as far as possible, MAYER's notation.

† Here and in the sequel the index i (and likewise j, k) is always understood to run from 1 to n, the index ρ from 1 to r.

$$(20) \qquad y_i = y_i(x), \qquad v_i = v_i(x), \qquad x_0 \leqq x \leqq x_1,$$

the equations (17) admit one and but one solution $y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_n$ for which the point

$$(x, y_1, \cdots, y_n, y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r, v_1, \cdots, v_n)$$

lies in the vicinity (ϵ) of the curve (17). We denote this solution by

$$(21) \qquad \begin{aligned} y_i' &= \Psi_i(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \\ \lambda_\rho &= \Pi_\rho(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \end{aligned}$$

so that, identically in (δ),

$$(22) \qquad \begin{aligned} \Omega_{n+i}(x, y_1, \cdots;\ \Psi_1, \cdots;\ \Pi_1, \cdots) &\equiv v_i, \\ f_\rho(x, y_1, \cdots;\ \Psi_1, \cdots) &\equiv 0, \end{aligned}$$

and the system of values $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ thus obtained lies in the interior of the region $\mathbf{T}$. In particular

$$(23) \qquad \begin{aligned} y_i'(x) &= \Psi_i[x, y_1(x), \cdots;\ v_1(x), \cdots], \\ \lambda_\rho(x) &= \Pi_\rho[x, y_1(x), \cdots;\ v_1(x), \cdots]. \end{aligned}$$

The first of these equations, combined with (19) and (I), shows that *the functions*

$$y_1(x), \cdots, y_n(x), v_1(x), \cdots, v_n(x)$$

satisfy the following system of $2n$ *differential equations:*

$$(\mathrm{II}) \qquad \begin{aligned} \frac{dy_i}{dx} &= \Psi_i(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \\ \frac{dv_i}{dx} &= \Omega_i(x, y_1, \cdots, y_n;\ \Psi_1, \cdots, \Psi_n;\ \Pi_1, \cdots, \Pi_n). \end{aligned}$$

These differential equations form a so-called *canonical system.* For, if we denote by

$$H(x, y_1, \cdots, y_n, v_1, \cdots, v_n)$$

that function of $x, y_1, \cdots, y_n, v_1, \cdots, v_n$ into which the expression

$$\sum_i y_i' \Omega_{n+i}(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots) - \Omega(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots)$$

is transformed by the substitution (21), i. e.,

$$(24)\qquad H(x, y_1, \cdots, v_1, \cdots) = \sum_i v_i\Psi_i - \Omega(x, y_1, \cdots; \Psi_1, \cdots; \Pi_1, \cdots),$$

and apply the identities (22), we easily obtain the result

$$\frac{\partial H}{\partial y_k} = -\,\Omega_k(x, y_1, \cdots; \Psi_1, \cdots; \Pi_1, \cdots),$$

$$\frac{\partial H}{\partial v_k} = \Psi_k(x, y_1, \cdots; v_1, \cdots).$$

Hence the system (II) may also be written

$$(\mathrm{II})\qquad \frac{dy_i}{dx} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dx} = -\frac{\partial H}{\partial y_i},$$

which is the characteristic form of a canonical system.

On account of (21_2), the expression for H may also be written

$$(24_a)\qquad H = \sum_i v_i\Psi_i - f(x, y_1, \cdots, \Psi_1, \cdots).$$

The functions on the right-hand side of (II) and the solution (20) of the canonical system (II) fulfil the conditions of lemma III. Hence we can take two quantities $X_0 < x_0$ and $X_1 > x_1$ so near to x_0 and x_1 respectively, and choose the positive quantity d so small that the following statements are true:

Let a^0 be an arbitrary value of x in the interval $(X_0 X_1)$ and write

$$y_i(a^0) = a_i^0, \qquad v_i(a^0) = b_i^0;$$

then if we choose

$$|a - a^0| \leqq d, \qquad |a_i - a_i^0| \leqq d, \qquad |b_i - b_i^0| \leqq d,$$

there exists a unique solution

$$(25)\qquad \begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) \\ v_i &= \mathfrak{V}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) \end{aligned} \qquad (X_0 \leqq x \leqq X_1)$$

of the system (II) having the following properties:

a) the functions $\mathfrak{Y}_i$, $\mathfrak{V}_i$, considered as functions of their $2n + 2$ arguments, as well as their derivatives indicated by the operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial a}, \frac{\partial}{\partial a_k}, \frac{\partial}{\partial b_k}, \frac{\partial^2}{\partial x\,\partial a}, \frac{\partial^2}{\partial x\,\partial a_k}, \frac{\partial^2}{\partial x\,\partial b_k}, \frac{\partial^2}{\partial x^2}{}^{*},$$

exist and are continuous throughout the domain

$$(26)\qquad X_0 \leqq x \leqq X_1, \ |a - a^0| \leqq d, \ |a_i - a_i^0| \leqq d, \ |b_i - b_i^0| \leqq d.$$

---

[*] The existence and continuity of $\partial^2\mathfrak{Y}_i/\partial x^2$, $\partial^2\mathfrak{Y}\,/\partial x^2$ follow from the fact that the functions on right-hand side of (II) are of class C' in (δ), which is slightly more than has been assumed in lemma III.

b) they satisfy the initial conditions

$$(27)\qquad \begin{aligned} \mathfrak{Y}_i(a;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) &= a_i, \\ \mathfrak{V}_i(a;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) &= b_i. \end{aligned}$$

c) the solution (25) lies in the vicinity (δ) of the curve (20).

d) for

$$a = a^0, \qquad a_i = a_i^0, \qquad b_i = b_i^0$$

the solution (25) coincides with the solution (20):

$$(28)\qquad \begin{aligned} y_i(x) &\equiv \mathfrak{Y}_i(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots), \\ v_i(x) &\equiv \mathfrak{V}_i(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots). \end{aligned}$$

e) the Jacobian

$$\frac{\partial(\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n, \mathfrak{V}_1, \cdots, \mathfrak{V}_n)}{\partial(a_1, \cdots, a_n, b_1, \cdots, b_n)}$$

is different from zero throughout the domain (26).

From the solution (25) of the system (II) a corresponding solution of the original system (I) is immediately obtained. For, from the definition of $\mathfrak{Y}_i$, $\mathfrak{V}_i$, it follows that *

$$(29)\qquad \mathfrak{Y}_i' = \Psi_i(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots);$$

hence if we define

$$(30)\quad \mathfrak{L}_\rho(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) = \Pi_\rho(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_n;\ \mathfrak{V}_1, \cdots, \mathfrak{V}_n),$$

it follows from (II_2) that

$$(31)\qquad \mathfrak{V}_i' = \Omega_i(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{L}_1, \cdots).$$

On the other hand, if in the identities (22) we replace y_i and v_i by $\mathfrak{Y}_i$ and $\mathfrak{V}_i$, respectively, and make use of (29) and (30) we obtain:

$$(32)\quad \Omega_{n+i}(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{L}_1, \cdots) \equiv \mathfrak{V}_i, \qquad f_\rho(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1, \cdots) \equiv 0.$$

Combining these equations with (31), we obtain the result:

The functions

$$(33)\qquad \begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n), \\ \lambda_\rho &= \mathfrak{L}_\rho(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n), \end{aligned}$$

satisfy Euler-Lagrange's differential equations (I) throughout the domain (26).

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* The accent always denotes differentiation with respect to x, even when other variables are present.

§ 4. *Hamilton's Principal Function.*

We consider now the integral

$$J=\int_{\mathfrak{E}} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\,dx$$

taken along the "extremal" *

$$\mathfrak{E}: \qquad y_i=\mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n)$$

from the point whose abscissa is a to the point whose abscissa is x.

We denote the value of this integral considered as a function of x, a, $a_1, \cdots, a$; $b_1, \cdots, b_n$ by

$$\mathfrak{U}(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n)$$

or simply $\mathfrak{U}$, so that

$$\mathfrak{U}=\int_a^x f(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{Y}_1', \cdots)\,dx, \tag{34}$$

or as we may also write on account of (32_2), (24_a) and (29):

$$\mathfrak{U}=\int_a^x \Omega(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{L}_1, \cdots)\,dx, \tag{34_a}$$

or

$$\mathfrak{U}=\int_a^x [\sum_i \mathfrak{V}_i\mathfrak{Y}_i' - H(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots)]\,dx. \tag{34_b}$$

We propose to compute the partial derivatives of $\mathfrak{U}$; for this purpose the expression (34_b) is most convenient.† We obtain immediately

$$\frac{\partial\mathfrak{U}}{\partial x}=\sum_i \mathfrak{V}_i\mathfrak{Y}_i' - H(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots). \tag{35}$$

Further, if we denote by α any one of the quantities a_k, b_k:

$$\frac{\partial\mathfrak{U}}{\partial\alpha}=\int_a^x \sum_i \left(\frac{\partial\mathfrak{V}_i}{\partial\alpha}\mathfrak{Y}_i' + \mathfrak{V}_i\frac{\partial\mathfrak{Y}_i'}{\partial\alpha} - H_i\frac{\partial\mathfrak{Y}_i}{\partial\alpha} - H_{n+i}\frac{\partial\mathfrak{V}_i}{\partial\alpha}\right)dx,$$

where

$$H_i=\frac{\partial H}{\partial y_i}, \qquad H_{n+i}=\frac{\partial H}{\partial v_i}, \tag{36}$$

and the arguments of H, H_i, H_{n+i} are x, $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n$, $\mathfrak{V}_1, \cdots, \mathfrak{V}_n$.

---

* We call *extremal* every set of functions $y_1, \cdots, y_n$ of x with which can be associated a system of functions $\lambda_1, \cdots, \lambda_r$ such that the functions y_i, λ_ρ constitute a solution of (I).

† With (34_a) the computation is slightly more complicated.

Applying to the second term under the integral sign integration by parts, we obtain

$$\frac{\partial \mathfrak{U}}{\partial \alpha} = \left[\sum_i \mathfrak{V}_i \frac{\partial \mathfrak{Y}_i}{\partial \alpha} \right]_a^x + \int_a^x \sum_i \left(\frac{\partial \mathfrak{V}_i}{\partial \alpha} (\mathfrak{Y}_i' - H_{n+i}) - \frac{\partial \mathfrak{Y}_i}{\partial \alpha} (\mathfrak{V}_i' + H_i) \right) dx .$$

But the functions $\mathfrak{Y}_i$, $\mathfrak{V}_i$ satisfy the system (II); hence the integral is zero and we obtain on account of (27):

$$\frac{\partial \mathfrak{U}}{\partial \alpha} = \sum_i \mathfrak{V}_i \frac{\partial \mathfrak{Y}_i}{\partial \alpha} - \sum_i b_i \left(\frac{\partial \mathfrak{Y}_i}{\partial \alpha} \right)_{x=a} .$$

Taking successively $\alpha = a_k$, b_k and making use of the relations

$$(37) \qquad \left(\frac{\partial \mathfrak{Y}_i}{\partial a_k} \right)_{x=a} = \delta_{ik} ,^* \qquad \left(\frac{\partial \mathfrak{Y}_i}{\partial b_k} \right)_{x=a} = 0 ,$$

derived from (27) by differentiation with respect to a_k, b_k, we obtain the result

$$(38) \qquad \frac{\partial \mathfrak{U}}{\partial a_k} = \sum_i \mathfrak{V}_i \frac{\partial \mathfrak{Y}_i}{\partial a_k} - b_k ,$$

$$(39) \qquad \frac{\partial \mathfrak{U}}{\partial b_k} = \sum_i \mathfrak{V}_i \frac{\partial \mathfrak{Y}_i}{\partial b_k} .$$

We now introduce, in addition to the assumptions A), B), C) of § 3, the new assumption concerning the solution (16) of the original system (I) that the Jacobian †

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* δ_{ik} is "KRONECKER's symbol," viz. $\delta_{ik} = 0$ if $i \neq k$, $\delta_{ii} = 1$.

† We notice incidentally that on account of (37_2) the Jacobian Δ is (apart from the sign) identical with MAYER's determinant

$$\Delta(x, a) = \begin{vmatrix} \frac{\partial \mathfrak{Y}_i}{\partial a_1}, & \cdots, & \frac{\partial \mathfrak{Y}_i}{\partial a_n}, & \frac{\partial \mathfrak{Y}_i}{\partial b_1}, & \cdots, & \frac{\partial \mathfrak{Y}_i}{\partial b_n} \\ \left(\frac{\partial \mathfrak{Y}_i}{\partial a_1} \right)_{x=a}, & \cdots, & \left(\frac{\partial \mathfrak{Y}_i}{\partial a_n} \right)_{x=a}, & \left(\frac{\partial \mathfrak{Y}_i}{\partial b_1} \right)_{x=a}, & \cdots, & \left(\frac{\partial \mathfrak{Y}_i}{\partial b_n} \right)_{x=a} \end{vmatrix} \qquad (i = 1, 2, \ldots, n),$$

which furnishes "JACOBI's criterion" for the problem under consideration [see A. MAYER, Journal für Mathematik, vol. 69 (1868), p. 250]. Hence it follows that it is possible to choose a^0 so that D) is satisfied, whenever "JACOBI's condition" is fulfilled (including the end-point x_1); see JORDAN, *Cours d'Analyse*, vol. 3, no. 393.

As to the possibility of an identical vanishing of the determinant $\Delta(x, a)$, compare MAYER, Journal für Mathematik, vol. 69, p. 251; *Encyclopædie*, II A, p. 599 (KNESER), p. 634 (ZERMELO and HAHN) and the references to VON ESCHERICH given there.

We also notice that the assumption D) implies that a^0 lies outside of the interval (x_0, x_1), since on account of (37_3)

$$\Delta(a^0; a^0, a_1^0, \cdots; b_1^0) = 0.$$

According to its definition, a^0 must therefore be taken in one of the two intervals

$$X_0 \leqq a^0 < x_0, \qquad \text{or} \qquad x_1 < a^0 \leqq X_1 .$$

$$\Delta(x;\ a,\ a_1,\ \cdots,\ a_n;\ b_1,\ \cdots,\ b_n)=\frac{\partial(\mathfrak{Y}_1,\ \cdots,\ \mathfrak{Y}_n)}{\partial(b_1,\ \cdots,\ b_n)}$$

is different from zero along the particular extremal

$$\mathfrak{E}_0: \qquad y_i=y_i(x)\equiv\mathfrak{Y}_i(x;\ a^0,\ a_1^0,\ \cdots;\ b_1^0,\ \cdots) \qquad (x_0\leqq x\leqq x_1),$$

i. e., that

$$D) \qquad \Delta(x;\ a^0,\ a_1^0,\ \cdots;\ b_1^0,\ \cdots)\neq 0 \qquad \text{for} \qquad x_0\leqq x\leqq x_1.$$

It follows then from our lemma III* that two positive quantities k and l ($l\leqq k$) can be assigned such that for every point $x,\ y_1,\ \cdots,\ y_n$ in the vicinity (l) of the extremal

$$\mathfrak{E}_0: \qquad y_i=y_i(x), \qquad x_0\leqq x\leqq x_1,$$

and for every $a,\ a_i$ such that

$$|a-a^0|<l, \qquad |a_i-a_i^0|<l,$$

the n equations

$$(40) \qquad y_i=\mathfrak{Y}_i(x;\ a,\ a_1,\ \cdots,\ a_n;\ b_1,\ \cdots,\ b_n)$$

admit one and but one solution $b_1,\ \cdots,\ b_n$ for which

$$|b_i-b_i^0|<k.$$

The corresponding inverse functions

$$(41) \qquad b_i=\mathfrak{B}_i(x,\ y_1,\ \cdots,\ y_n:\ a,\ a_1,\ \cdots,\ a_n)$$

are of class C' in the domain

$$(42) \qquad (x,\ y_1,\ \cdots,\ y_n)\ \text{in}\ (l)_{\mathfrak{E}_0}; \qquad |a-a^0|<l, \qquad |a_i-a_i^0|<l;$$

they satisfy in this domain the identities

$$(43) \qquad \mathfrak{Y}_i(x;\ a,\ a_1,\ \cdots;\ \mathfrak{B}_1,\ \cdots)\equiv y_i.$$

If we replace in $\mathfrak{U}(x;\ a,\ a_1,\ \cdots;\ b_1,\ \cdots)$ the b_i's by the function $\mathfrak{B}_i$, the function $\mathfrak{U}$ is transformed into a function of $x,\ y_1,\ \cdots,\ y_n;\ a,\ a_1,\ \cdots,\ a_n$; we denote it by

$$(44)\ \mathfrak{W}(x,\ y_1,\ \cdots,\ y_n;\ a,\ a_1,\ \cdots,\ a_n)\equiv\mathfrak{U}(x;\ a,\ a_1,\ \cdots,\ a_n;\ \mathfrak{B}_1,\ \cdots,\ \mathfrak{B}_n).$$

It represents *the value of the integral J taken along the uniquely defined extremal joining the two points $(a,\ a_1,\ \cdots,\ a_n)$ and $(x,\ y_1,\ \cdots,\ y_n)$, considered as a function of the coördinates of these two points.* It is identical with

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* To the point set **C** of the lemma corresponds here the set

$$x_0\leqq x\leqq x_1,\ y_i=y_i(x),\ a=a^0,\ a_i=a_i^0,\ b_i=b_i^0.$$

HAMILTON's *Principal Function*[*] and a generalization of DARBOUX's *Geodesic Distance.*[†]

From the identities (43) we obtain by differentiation the relations

$$
(45) \qquad
\begin{aligned}
&\overline{\frac{\partial \mathfrak{Y}_i}{\partial a_k}} + \sum_j \overline{\frac{\partial \mathfrak{Y}_i}{\partial b_j}}\, \frac{\partial \mathfrak{B}_j}{\partial a_k} = 0, \\
&\overline{\frac{\partial \mathfrak{Y}_i}{\partial x}} + \sum_j \overline{\frac{\partial \mathfrak{Y}_i}{\partial b_j}}\, \frac{\partial \mathfrak{B}_j}{\partial x} = 0, \\
&\sum_j \overline{\frac{\partial \mathfrak{Y}_i}{\partial b_j}}\, \frac{\partial \mathfrak{B}_j}{\partial y_k} = \delta_{ik},
\end{aligned}
$$

where we have indicated by a stroke the substitution of $\mathfrak{B}_i$ for b_i.

If we combine the relations (45) with the previous results (35), (38), (39), we obtain the following theorems due to HAMILTON:[*]

The partial derivatives of Hamilton's "Principal Function," defined by the equations (34) *and* (44), *have the following values*

$$
(46) \qquad
\begin{aligned}
&\frac{\partial \mathfrak{W}}{\partial x} = -H(x, y_1, \cdots, y_n;\ \bar{\mathfrak{B}}_1, \cdots, \bar{\mathfrak{B}}_n), \\
&\frac{\partial \mathfrak{W}}{\partial y_k} = \bar{\mathfrak{B}}_k, \qquad \frac{\partial \mathfrak{W}}{\partial a_k} = -\mathfrak{B}_k,
\end{aligned}
$$

where

$$\bar{\mathfrak{B}}_k = \mathfrak{B}_k(x;\ a, a_1, \cdots, a_n;\ \mathfrak{B}_1, \cdots, \mathfrak{B}_n).$$

An immediate consequence (which we shall however not need in the sequel) is the well-known theorem that $\mathfrak{W}$ satisfies "HAMILTON's partial differential equation":

$$\frac{\partial \mathfrak{W}}{\partial x} + H\left(x, y_1, \cdots, y_n, \frac{\partial \mathfrak{W}}{\partial y_1}, \cdots, \frac{\partial \mathfrak{W}}{\partial y_n}\right) = 0.\text{‡}$$

§ 5. *Weierstrass' theorem for a set of extremals through a fixed point.*

We now give the quantities a, a_i the special values

$$a = a^0,\ a_i = a_i^0,$$

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[*] Philosophical Transactions, 1835, part I, p. 99.

[†] *Theorie des Surfaces*, vol. 2, no. 536.

[‡] All these results might have been taken directly from HAMILTON's theory, for instance in E. VON WEBER's presentation in his *Vorlesungen über das Pfaff'sche Problem*, art. 386. But on the one hand, I did not wish to presuppose HAMILTON's theory, on the other hand the question of the inverse functions $\mathfrak{B}_i$ requires a slightly different treatment from the Calculus of Variations standpoint.

and write for brevity

$$\mathfrak{Y}_i(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{y}_i(x, b_1, \cdots, b_n),$$

$$\mathfrak{B}_i(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{v}_i(x, b_1, \cdots, b_n),$$

$$\mathfrak{L}_\rho(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{l}_\rho(x, b_1, \cdots, b_n).$$

Then the equations

$$y_i = \mathfrak{y}_i(x;\ b_1, \cdots, b_n), \tag{47}$$

in which $b_1, \cdots, b_n$ are considered as variable parameters, represent the set of extremals through the point

$$x = a^0,\ y_1 = a_1^0, \cdots, y_n = a_n^0,$$

and from what has been said in § 4 about the solution of the equations (40), it follows a fortiori, that for every $x, y_1, \cdots, y_n$ in the vicinity $(l)_{\mathfrak{E}_0}$ the equations (47) admit one and but one solution $b_1, \cdots, b_n$ for which $|b_i - b_i^0| < k$. The vicinity $(l)_{\mathfrak{E}_0}$ constitutes therefore a *field of extremals about the extremal* $\mathfrak{E}_0$.

The corresponding inverse functions are

$$b_i = \mathfrak{B}_i(x, y_1, \cdots, y_n, a^0, a_1^0, \cdots, a_n^0) \equiv \mathfrak{b}_i(x, y_1, \cdots, y_n).$$

Further we define

$$\mathfrak{w}(x, y_1, \cdots, y_n) = \mathfrak{W}(x, y_1, \cdots, y_n;\ a^0, a_1^0, \cdots, a_n^0). \tag{48}$$

The function $\mathfrak{w}(x, y_1, \cdots, y_n)$ is identical with the value of the integral J taken from the fixed point $(a^0, a_1^0, \cdots, a_n^0)$ to the point $(x, y_1, \cdots, y_n)$ of the field along the unique extremal of the field passing through the point $(x, y_1, \cdots, y_n)$; we shall call it the *field integral*. From (46) it follows that its partial derivatives have the values

$$\frac{\partial \mathfrak{w}}{\partial x} = -H(x, y_1, \cdots, y_n;\ \bar{\mathfrak{v}}_1, \cdots, \bar{\mathfrak{v}}_n), \qquad \frac{\partial \mathfrak{w}}{\partial y_k} = \bar{\mathfrak{v}}_k, \tag{49}$$

where the stroke indicates the substitution of $\mathfrak{b}_i$ for b_i. These expressions may be thrown into a different form by introducing the functions

$$\begin{aligned} p_i(x, y_1, \cdots, y_n) &= \mathfrak{y}_i'(x;\ \mathfrak{b}_1, \cdots, \mathfrak{b}_n), \\ \mu_\rho(x, y_1, \cdots, y_n) &= \mathfrak{l}_\rho(x;\ \mathfrak{b}_1, \cdots, \mathfrak{b}_n). \end{aligned} \tag{50}$$

For, according to (29), (30) and (32),

$$\mathfrak{v}_i = \Omega_{n+i}(x, \mathfrak{y}_1, \cdots;\ \mathfrak{y}_i', \cdots;\ \mathfrak{l}_1, \cdots),$$

$$\mathfrak{y}_i' = \Psi_i(x, \mathfrak{y}_1, \cdots;\ \mathfrak{v}_1, \cdots),$$

$$\mathfrak{l}_\rho = \Pi_\rho(x, \mathfrak{y}_1, \cdots;\ \mathfrak{v}_1, \cdots)$$

and therefore on replacing b_i by $\mathfrak{b}_i$, and applying (50), we get

$$\begin{aligned}\bar{\mathfrak{v}}_i &= \Omega_{n+i}(x, y_i, \cdots; p_1, \cdots; \mu_1, \cdots),\\ p_i &= \Psi_i(x, y_1, \cdots; \bar{\mathfrak{v}}_1, \cdots),\\ \mu_\rho &= \Pi_\rho(x, y_1, \cdots; \bar{\mathfrak{v}}_1, \cdots).\end{aligned}$$

Substituting finally for H its value from (24_a), we obtain the result:

The partial derivatives of the field-integral $\mathfrak{w}$ *have the following values:*

$$(51)\qquad \begin{aligned}\frac{\partial \mathfrak{w}}{\partial x} &= f(x, y_1, \cdots; p_1, \cdots) - \sum_i p_i \Omega_{n+i}(x, y_1, \cdots; p_1, \cdots; \mu_1 \cdot \cdots),\\ \frac{\partial \mathfrak{w}}{\partial y_k} &= \Omega_{n+k}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots).\end{aligned}$$

Let now

$$\tilde{\mathfrak{C}}: \qquad y_i = \tilde{y}_i(x), \qquad x_0 \leqq x \leqq x_1$$

be any curve of class C' which satisfies the differential equations (5), lies in the field $(l)_{\mathfrak{E}_0}$, and passes through the two end points of $\mathfrak{E}_0$, viz.,

$$P_0(x_0, y_{10}, \cdots, y_{n0}) \qquad \text{and} \qquad P_1(x_1, y_{11}, \cdots, y_{n1}),$$

where

$$y_{i0} = y_i(x_0), \qquad y_{i1} = y_i(x_1).$$

To this curve we can apply WEIERSTRASS' *construction*. In order to fix the ideas we choose* $a^0 < x_0$. Then, if $P(x, y_1, \cdots, y_n)$ is any point of $\tilde{\mathfrak{C}}$, we draw the unique extremal $\mathfrak{E}$ of the field from P^0 to P and consider the function

$$S(x) = J_{\mathfrak{E}}(P^0P) + J_{\tilde{\mathfrak{C}}}(PP_1).$$

We have then, in the usual manner,

$$\Delta J = J_{\tilde{\mathfrak{C}}} - J_{\mathfrak{E}_0} = -[S(x_1) - S(x_0)].$$

But

$$J_{\mathfrak{E}}(P^0P) = \mathfrak{w}[x, \tilde{y}(x), \cdots, \tilde{y}_n(x)],$$

$$J_{\tilde{\mathfrak{C}}}(PP_1) = \int_x^{x_1} f[x, \tilde{y}_1(x), \cdots, \tilde{y}_n(x), \tilde{y}_1'(x), \cdots, \tilde{y}_n'(x)]\,dx.$$

Hence we obtain for the derivative $S'(x)$:

$$S'(x) = \frac{\partial \mathfrak{w}}{\partial x} + \sum_i \frac{\partial \mathfrak{w}}{\partial y_i}\tilde{y}_i' - f(x, y_1, \cdots, y_n, \tilde{y}_1', \cdots, \tilde{y}_n'),$$

that is, according to (51),

$$\begin{aligned}S'(x) = -\sum_i (p_i - \tilde{y}_i')\Omega_{n+i}(x, y_1, \cdots, p_1, \cdots, \mu_1, \cdots)&\\ + f(x, y_1, \cdots, p_1, \cdots) - f(x, y_1, \cdots, \tilde{y}_1', \cdots)&.\end{aligned}$$

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* Compare the footnote on p. 473.

Hence if we introduce the E-function by the definition

$$\begin{aligned} E(x, y_1, \cdots, y_n;\ p_1, \cdots, p_n;\ \tilde{y}'_1, \cdots, \tilde{y}'_n;\ \mu_1, \cdots, \mu_r) = f(x, y_1, \cdots, \tilde{y}'_1, \cdots) \\ - f(x, y_1, \cdots, p_1, \cdots) - \sum_i (\tilde{y}'_i - p_i)\, \Omega_{n+i}(x, y_1, \cdots, p_1, \cdots, \mu_1, \cdots), \end{aligned} \tag{52}$$

we obtain *Weierstrass' Theorem*:

$$\begin{aligned} \Delta J = \int_{x_0}^{x_1} E\{x, \tilde{y}_1(x), \cdots;\ p_1[x, y_1(x), \cdots], \cdots; \\ \tilde{y}'_1(x)\ \ \cdots;\ \mu_1[x, \tilde{y}_1(x), \cdots], \cdots\} dx, \end{aligned} \tag{53}$$

in which the functions p_i, μ_ρ are defined by (50).

§ 6. *Kneser's Problem generalized.*

In the preceding section WEIERSTRASS' theorem has been proved for the special case of an n-parametric set of extremals passing through a fixed point; in this and the following sections, we propose to extend the theorem to a more general class of n-parametric sets. The method which we shall use will be entirely analogous to the method by which KNESER* has accomplished the same result for the simplest case $n = 1$, $r = 0$.

Accordingly, let

$$y_i = Y_i(x, a_1, \cdots, a_n), \qquad v_i = V_i(x, a_1, \cdots, a_n), \tag{54}$$

be any n-parametric set of solutions of the differential equations (II), containing the special solution

$$y_i = y_i(x), \qquad v_i = v_i(x), \tag{20}$$

say for $a_i = a_i^0$, and furnishing a field† S_k of extremals about the particular extremal $\mathfrak{E}_0$. The inverse functions of the field will be denoted by

$$a_i = \mathfrak{a}_i(x, y_1, \cdots, y_n)$$

so that identically

$$Y_i(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}_n) \equiv y_i. \tag{56}$$

Across the field S_k we draw an n-dimensional surface which meets every extre-

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* Compare the Introduction.

† We include in this assumption the condition that the functions Y_i, $\partial Y_i / \partial x$ shall be of class C' in a certain domain

$$x_0 - \delta < x < x_1 + \delta, \qquad |a_i - a_i^0| < \delta.$$

The field S_k consists of all points x, y_1, $\cdots$, y_n furnished by the equations

$$y_i = Y_i(x;\ a_1, \cdots, a_n), \tag{55}$$

when x, a_1, $\cdots$, a_n are restricted to the domain

$$x_0 - k < x < x_1 + k, \qquad |a_i - a_i^0| < k.$$

mal of the field in one and but one point; such a surface may be represented, in parameter-representation, in the form

$$(57)\qquad x=\xi(a_1,\cdots,a_n),\qquad y_i=Y_i(\xi;\,a_1,\cdots,a_n).$$

Then we consider our integral J taken along the extremal $\mathfrak{E}:a_1,\cdots,a_n$ of the set (55) from its point of intersection with the surface (57), i. e., from the point P^0 whose abscissa is $\xi(a_1,\cdots,a_n)$, to the point P whose abscissa is x, i. e., the integral

$$(58)\qquad U(x;\,a_1,\cdots,a_n)=\int_{\xi(a_1,\,\ldots,\,a_n)}^{x} f(x,\,Y_1,\,\cdots,\,Y_1',\,\cdots)dx.$$

The same integral, considered as a function of the coördinates $x,\,y_1,\,\cdots,\,y_n$ of the end point P will be denoted by $W(x,\,y_1,\,\cdots,\,y_n)$, so that

$$(59)\qquad W(x,\,y_1,\,\cdots,\,y_n)=U(x;\,\mathfrak{a}_1,\,\cdots,\,\mathfrak{a}_n).$$

We call the surface (57) the "surface of reference" for the definition of the function W.

And now *we propose so to choose the surface* (57) *i. e., the function* $\xi(a_1,\cdots,a_n)$, *that the partial derivatives of the function* W *shall have the same simple form as in the previous case of a set of extremals through a fixed point, viz.,*

$$(60)\qquad \begin{aligned}\frac{\partial W(x,\,y_1,\,\cdots,\,y_n)}{\partial x}&=-H(x,\,y_1,\,\cdots,\,y_n,\,\bar{V}_1,\,\cdots,\,\bar{V}_n),\\ \frac{\partial W(x,\,y_1,\,\cdots,\,y_n)}{\partial y_k}&=\bar{V}_k,\end{aligned}$$

where

$$\bar{V}_k=V_k(x;\,\mathfrak{a}_1,\,\cdots,\,\mathfrak{a}_n).$$

As soon as this is accomplished **Weierstrass'** theorem for the set (55) will follow immediately.

From the connection between the functions $W(x,\,y_1,\,\cdots,\,y_n)$ and $U(x;\,a_1,\,\cdots,\,a_n)$ it follows that the partial derivatives of W will take the above form if and only if the partial derivatives of U have the values

$$(61)\qquad \frac{\partial U}{\partial x}=f(x,\,Y_1,\,\cdots,\,Y_1',\,\cdots),\qquad \frac{\partial U}{\partial a_k}=\sum_i V_i\frac{\partial Y}{\partial a_k},$$

corresponding to (35) and (39).

The expression for $\partial U/\partial x$ has always the desired form, however we may choose ξ; it remains, therefore, so to determine the function $\xi(a_1,\cdots,a_n)$ that (61_2) shall hold.

We start from the remark, which is easily verified, that every n-parametric set of solutions of the system (II), which furnishes a field of extremals about our extremal $\mathfrak{E}_0$, can be thrown by a parameter-transformation, into the canonical form

$$(62)\qquad \begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n),\\ v_i &= \mathfrak{V}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n), \end{aligned}$$

where $B_1, \cdots, B_n$ are functions of $a_1, \cdots, a_n$ of class C' in a certain vicinity of $a_1 = a_1^0, \cdots, a_n = a_n^0$, satisfying the initial conditions

$$(63)\qquad B_i(a_1^0, \cdots, a_n^0) = b_i^0;$$

the quantities $a^0, a_1^0, \cdots, a_n^0$ have the same signification as in § 3, and $a_1, \cdots, a_n$ are considered as the parameters of the set.

We may, without loss of generality, suppose that the parameters $a_1, \cdots, a_n$ of the set (54) are these canonical parameters, so that

$$(64)\qquad \begin{aligned} Y_i(x;\ a_1, \cdots, a_n) &\equiv \mathfrak{Y}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n),\\ V_i(x;\ a_1, \cdots, a_n) &\equiv \mathfrak{V}_i(a;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n), \end{aligned}$$

whence it follows that

$$Y_i(a^0;\ a_1, \cdots, a_n) = a_i, \qquad V_i(a^0;\ a_1, \cdots, a_n) = B_i.$$

The integral $U(x;\ a_1, \cdots, a_n)$ may therefore be expressed in terms of the function $\mathfrak{U}$ defined by (34), viz:

$$U(x;\ a_1, \cdots, a_n) = \mathfrak{U}(x;\ a^0, a_1, \cdots;\ B_1, \cdots) - \mathfrak{U}(\xi;\ a^0, a_1, \cdots;\ B_1, \cdots).$$

Hence we obtain

$$\frac{\partial U}{\partial a_k} = \left(\frac{\partial \mathfrak{U}(x)}{\partial a_k}\right) + \sum_j \left(\frac{\partial \mathfrak{U}(x)}{\partial b_j}\right)\frac{\partial B_j}{\partial a_k} - \frac{\partial \mathfrak{U}(\xi)}{\partial a_k},$$

where the bracket ( ) indicates the substitution of B_i for b_i and of a^0 for a. Substituting the values of $\partial \mathfrak{U}/\partial a_k$, $\partial \mathfrak{U}/\partial b_k$ from (38) and (39), and noticing that according to (64)

$$\frac{\partial Y_i}{\partial a_k} = \left(\frac{\partial \mathfrak{Y}_i}{\partial a_k}\right) + \sum_j \left(\frac{\partial \mathfrak{Y}_i}{\partial b_j}\right)\frac{\partial B_j}{\partial a_k},$$

we obtain

$$(65)\qquad \frac{\partial U}{\partial a_k} = \sum_i V_i \frac{\partial Y_i}{\partial a_k} - B_k - \frac{\partial \mathfrak{U}(\xi)}{\partial a_k}.$$

In order that $\partial U/\partial a_k$ may have the desired form (61_2), it is therefore necessary and sufficient that

$$(66)\qquad B_k = -\frac{\partial \mathfrak{U}(\xi)}{\partial a_k}.$$

Hence we infer in the first place: *The desired determination of the function $\xi(a_1, \cdots, a_n)$ is, for $n > 1$, not possible for all n-parametric sets of ex-*

tremals, but only for those special sets for which in the normal form (62) *the functions* $B_1, \cdots, B_n$ *are the partial derivatives with respect to* $a_1, \cdots, a_n$ *respectively of one and the same function of* $a_1, \cdots, a_n$, *for which therefore the functions* B_i *satisfy the integrability condition*

$$\frac{\partial B_h}{\partial a_k} = \frac{\partial B_k}{\partial a_h}. \tag{67}$$

We shall call these special n-parametric sets of extremals *Mayerian sets of extremals*, because they are identical with the sets discovered by MAYER,* which have the peculiarity of furnishing solutions of HILBERT's Problem.

We suppose in the sequel that for our set (54) the condition (67) is fulfilled, so that

$$B_k = \frac{\partial A(a_1, \cdots, a_n)}{\partial a_k}, \tag{68}$$

where $A(a_1, \cdots, a_n)$ is any function of $a_1, \cdots, a_n$, of class C'' in the vicinity of $a_1^0, \cdots, a_n^0$, which satisfies according to (63) the initial condition

$$\left.\frac{\partial A(a_1, \cdots, a_n)}{\partial a_k}\right|^{a_i = a_i^0} = b_k^0. \tag{69}$$

Our condition (66) takes then the form

$$\mathfrak{U}\left(\xi;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right) = -A(a_1, \cdots, a_n) + c, \tag{70}$$

c being a numerical constant independent of $a_1, \cdots, a_n$.

For the discussion of this equation we write for brevity

$$\mathfrak{U}\left(x;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right) + A(a_1, \cdots, a_n) \equiv G(x;\ a_1, \cdots, a_n)$$

and add to our previous assumptions† A) to D) the further assumption

E) $\qquad f[x, y_1(x), \cdots;\ y_1'(x), \cdots] \neq 0$ in $(x_0 x_1)$.

Under this assumption we consider the problem of solving the equation

$$G(x;\ a_1, \cdots, a_n) = u \tag{71}$$

with respect to x.

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* See MAYER II, §§ 2, 3; compare also below, § 8, end. From the fact that for the special sets of extremals through a fixed point the partial derivatives of the field integral have the simple form (49), it follows that these special sets, when properly normalized, satisfy the condition (67). Compare MAYER's remarks on this point, MAYER, II, p. 63.

† See pp. 468 and 474.

Since according to (69), (64), and (28)

$$Y_i(x;\ a_1^0, \cdots, a_n^0) \equiv y_i(x),$$

it follows from E) that

$$\frac{\partial G(x;\ a_1, \cdots, a_n)}{\partial x} \neq 0$$

for

$$x_0 \leqq x \leqq x_1, \qquad a_i = a_i^0,$$

and therefore

$$G(x';\ a_1^0, \cdots, a_n^0) \neq G(x'';\ a_1^0, \cdots, a_n^0),$$

if x', x'' are any two distinct values of x in the interval $(x_0 x_1)$. Hence on applying * lemma I of § 1 to the system of equations

$$u = G(x;\ a_1, \cdots, a_n), \qquad u_i = a_i,$$

we obtain the result that *the equation* (71) *admits a unique solution*

$$(72) \qquad x = \xi(a_1, \cdots, a_n;\ u)$$

in the following sense: The positive quantity k can be taken so small that if x', a_1', $\cdots$, a_n' is any point in the domain

$$x_0 - k < x < x_1 + k, \qquad |a_i - a_i^0| < k,$$

and $G(x';\ a_1', \cdots, a_n') = u'$, then the equation

$$G(x;\ a_1', \cdots, a_n') = u'$$

has no other solution than $x = x' = \xi(a_1', \cdots, a_n';\ u')$, satisfying the condition

$$x_0 - k < x < x_1 + k.$$

Further, if G_0 and G_1 denote the minimum and maximum of the function $G(x;\ a_1^0, \cdots, a_n^0)$ in the interval $(x_0 x_1)$, it follows from the corollary to lemma I that a second positive quantity $l \leqq k$ can be determined so that for every $a_1, \cdots, a_n;\ u$ in the domain

$$(73) \qquad G_0 - l < u < G_1 + l, \qquad |a_i - a_i^0| < l,$$

equation (71) has one and but one solution

$$x = \xi(a_1, \cdots, a_n;\ u)$$

for which $x_0 - k < x < x_1 + k$. Moreover the equation

$$(74) \qquad G[\xi(a_1, \cdots, a_n;\ u), a_1, \cdots, a_n] = u$$

---

* To the point set C of the lemma corresponds here the set

$$x_0 \leqq x \leqq x_1, \quad a_i = a_i^0.$$

holds identically in the domain (73), and the function $\xi(a_1, \cdots, a_n; u)$ is of class C' in the same domain.

Hence follows the result: *If the functions* B_i *satisfy the condition* (67), *our problem has an infinitude of solutions, viz.*,

$$\xi = \xi(a_1, \cdots, a_n; c),$$

differing from each other by the value of the constant c. The latter may take any value between $G_0 - l$ and $G_1 + l$. If we take $\xi(a_1, \cdots, a_n; c)$ as lower limit in the definition of the function $U(x; a_1, \cdots, a_n)$, the partial derivatives of U have the desired values (61). The functions U corresponding to two different values c' and c'' of the constant c can therefore differ only by an additive constant independent of $x, a_1, \cdots, a_n$, a result which can immediately be verified directly, since, according to (74),

$$\begin{aligned}(75)\qquad \int_{\xi(a_1, \ldots, a_n; c')}^{\xi(a_1, \ldots, a_n; c'')} & f(x, Y_1, \cdots; Y_1', \cdots)dx \\ &= G[\xi(a_1, \cdots, a_n; c''); a_1, \cdots, a_n] \\ &\quad - G[\xi(a_1, \cdots, a_n; c'); a_1, \cdots, a_n] = c'' - c'.\end{aligned}$$

§ 7 *Kneser's theorem on transversals generalized.*

Passing now from the consideration of the function $U(x, a_1, \cdots, a_n)$ to the function $W(x, y_1, \cdots, y_n)$, we write for brevity

$$(76)\qquad Y_i[\xi(a_1, \cdots, a_n; u); a_1, \cdots, a_n] \equiv \eta_i(a_1, \cdots, a_n; u).$$

The equations

$$(77)\qquad x = \xi(a_1, \cdots, a_n; c), \qquad y_i = \eta_i(a_1, \cdots, a_n; c)$$

represent then, in parameter-representation, an n-dimensional surface $\mathfrak{T}_c$ in the $(n+1)$-dimensional space of the variables $x, y_1, \cdots, y_n$. It lies entirely in the field $\mathbf{S}_k$ if $a_1, \cdots, a_n$ are restricted to the domain: $|a_i - a_i^0| < l$; giving c different constant values we obtain a one-parameter set of surfaces, and from what has been said about the solution of the equation (71), it follows that through every point of the field $\mathbf{S}_k$ passes one and but one surface of the set.

We shall call these surfaces the "*transversal surfaces*" of the field $\mathbf{S}_k$ formed by the set of extremals

$$(55)\qquad y_i = Y_i(x; a_1, \cdots, a_n),$$

because they are the generalization of KNESER's "transversals." If we take the integral $W(x, y_1, \cdots, y_n)$ from the point of intersection of the extremal passing through the point $(x, y_1, \cdots, y_n)$ with a fixed transversal surface of reference, the partial derivatives of W have the desired form (60). If we change the transversal surface of reference, the function $W(x, y_1, \cdots, y_n)$ changes only by an additive constant; the function $W(x, y_1, \cdots, y_n)$ is therefore determined up to an additive constant by the set of extremals (55) forming the field; we shall call it the *field-integral.*

Exactly as in § 5, *the expressions* (60) *for the partial derivatives of the field integral may be thrown into the form*

$$(79)\qquad \begin{aligned} \frac{\partial W}{\partial x} &= f(x, y_1, \cdots; p_1, \cdots) - \sum_i p_i \Omega_{n+i}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots), \\ \frac{\partial W}{\partial y_k} &= \Omega_{n+k}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots), \end{aligned}$$

the functions p_i, μ_ρ being defined by the equations

$$(79)\qquad \begin{aligned} p_i(x, y_1, \cdots, y_n) &= Y_i'(x; \mathfrak{a}_1, \cdots, \mathfrak{a}_n) \\ \mu_\rho(x, y_1, \cdots, y_n) &= \Lambda_\rho(x; \mathfrak{a}_1, \cdots, \mathfrak{a}_n), \end{aligned}$$

where in accordance with (64)

$$\Lambda_\rho(x, a_1, \cdots, a_n) = \mathfrak{L}_\rho(x;\ a^0, a_1^0, \cdots, a_n^0;\ B_1, \cdots, B_n).$$

The equation (75) interpreted for the function $W(x, y_1, \cdots, y_n)$ contains the *extension of Kneser's fundamental theorem on transversals:*

Two transversal surfaces $\mathfrak{T}_{c'}$, $\mathfrak{T}_{c''}$ of the same field, formed by a Mayerian set of extremals, intercept on the extremals of the field arcs along which the integral J has a constant value, viz., $c'' - c'$.

Conversely: If the surface of reference for the integral $W(x, y_1, \cdots y_n)$ is the transversal surface $\mathfrak{T}_{c'}$, then the points of the field for which

$$W(x, y_1, \cdots, y_n) = c'' - c'$$

lie on the transversal surface $\mathfrak{T}_{c''}$.

Hence the transversal surfaces (77) are identical with the surfaces

$$W(x, y_1, \cdots, y_n) = \text{const.}$$

To complete the analogy with the case $n = 1$, $r = 0$, it remains to show that the transversal surfaces can also be characterized by differential equations which are the analogue of KNESER's condition of transversality. Differentiating the identity (74) with respect to a_k and utilizing the results obtained in deriving (65), we get *

$$(80)\qquad f(x, Y_1, \cdots, Y_1', \cdots)\frac{\partial x}{\partial a_k} + \sum_i V_i \frac{\partial Y_i}{\partial a_k}\Bigg|^{x=\xi} = 0.$$

---

* The substitution symbol $|^{x=\xi}$ refers to the whole left-hand side of equation (80).

Conversely, if ξ is a function of $a_1, \cdots, a_n$ satisfying the n partial differential equations (80), then

$$G(\xi;\ a_1, \cdots, a_n) = \text{const.},$$

i. e., independent of $a_1, \cdots, a_n$.

Hence the transversal surfaces may also be characterized by the n partial differential equations (80) for the function ξ; these differential equations are compatible in consequence of the relation (67).

If we introduce the functions η_i defined by (76), we have

$$Y'_i \frac{\partial x}{\partial a_k} + \frac{\partial Y_i}{\partial a_k}\Big|^{x=\xi} = \frac{\partial \eta_i}{\partial a_k},$$

and (80) may be written

$$[f(x, Y_1, \cdots, Y'_1, \cdots) - \sum_i Y'_i V_i]\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i V_i \Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0. \tag{81}$$

On account of (24_a), (29) and (32), the same equation may also be thrown into one of the following two forms

$$-H(x, Y_1, \cdots;\ V_1, \cdots)\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i V_i \Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0, \tag{81_a}$$

$$(\Omega - \sum_i Y'_i \Omega_{n+i})\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i \Omega_{n+1}\Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0, \tag{81_b}$$

the arguments of Ω, Ω_{n+i} being x, $Y_1, \cdots, Y'_1, \cdots, \Lambda_1, \cdots$.

Condition (81), in any one of its three equivalent forms, is the analogue of *Kneser's condition of transversality*; for, in the simplest case $n = 1$, $r = 0$, the n equations (81_b) reduce to the one equation

$$[f(x, Y, Y') - Y' f_{y'}(x, Y, Y')]\Big|^{x=\xi} \frac{\partial \xi}{\partial a} + f_{y'}(x, Y, Y')\Big|^{x=\xi} \frac{\partial \eta}{\partial a} = 0,$$

which is the well-known condition of transversality.

§ 8. *Weierstrass' theorem for a Mayerian set of extremals.*

The fundamental formulas (78) being once established, WEIERSTRASS' theorem follows immediately by *Kneser's modification of Weierstrass' construction.**

For, let $\mathfrak{T}_0$ be the transversal surface through the end-point P_0 of the extremal $\mathfrak{E}_0$, and $\mathfrak{T}'$ a second transversal surface of the field which meets the continuation of $\mathfrak{E}_0$ at a point P'_0 whose abscissa is less than x_0. From an arbitrary point P_2 of $\mathfrak{T}_0$ we draw to the second end-point P_1 of $\mathfrak{E}_0$ any curve

$$\tilde{\mathfrak{E}}: \qquad y_i = \tilde{y}_i(x)$$

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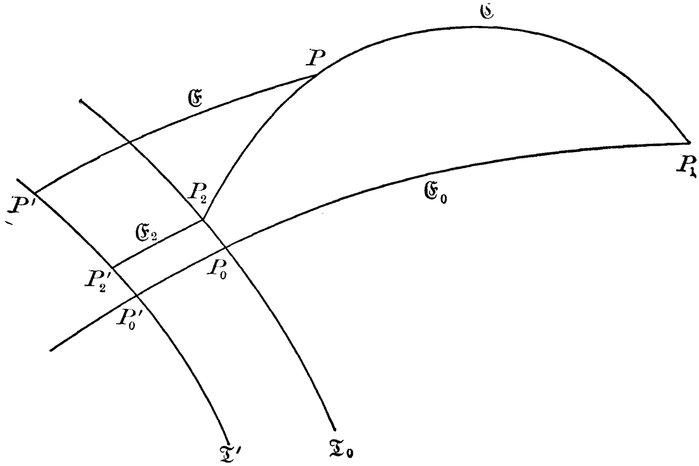
*See KNESER, *Lehrbuch*, § 20, and BOLZA, *Lectures*, § 37, *a*).

of class C', lying wholly* in the field $\mathbf{S}_k$ and satisfying the differential equations

$$f_\rho(x, \tilde{y}_1(x), \cdots, \tilde{y}_1'(x), \cdots) = 0.$$



Through an arbitrary point $P(x, y_1, \cdots, y_n)$ of $\tilde{\mathfrak{E}}$ passes a unique extremal $\mathfrak{E}$ of the field, which meets the transversal surface $\mathfrak{T}'$ at one and but one point P'. We consider then the function

$$S(x) = J_{\mathfrak{E}}(P'P) + J_{\tilde{\mathfrak{E}}}(PP_1).$$

Since according to the generalized theorem on transversals,

$$J_{\mathfrak{E}_2}(P_2'P_2) = J_{\mathfrak{E}_0}(P_0'P_0),$$

(see figure), it follows exactly as in the case $n = 1$, $r = 0$, that

$$\Delta J \equiv J_{\tilde{\mathfrak{E}}}(P_2P_1) - J_{\mathfrak{E}_0}(P_0P_1) = \int_{x_2}^{x_1} S'(x)\,dx.$$

But

$$J_{\mathfrak{E}}(P'P) = W[x, \tilde{y}_1(x), \cdots, \tilde{y}_n(x)],$$

if $\mathfrak{T}'$ is used as surface of reference for W; and since the expressions for the partial derivatives of W are the same as in the special case of a set of extremals through a fixed point, also the computation of $S'(x)$ is the same as in § 5 and need therefore not be repeated here. We thus obtain WEIERSTRASS' theorem in the following form:

Let

$$y_i = Y_i(x;\ a_1, \cdots, a_n) \equiv \mathfrak{Y}_i\left(x;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right),$$

$$\lambda_\rho = \Lambda_\rho(x;\ a_1, \cdots, a_n) \equiv \mathfrak{L}_\rho\left(x;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right),$$

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* Notice that the field $\mathbf{S}_k$ is a continuum, as follows from lemma I in § 1.

be a Mayerian n-parametric set of solutions of Euler-Lagrange's differential equations (I) *furnishing a field of extremals,* $\mathbf{S}_k$, *about the particular extremal* $\mathfrak{E}_0$. *Let further*

$$a_i = \mathfrak{a}_i(x, y_1, \cdots, y_n)$$

denote the inverse functions of the field and define

$$p_i(x, y_1, \cdots, y_n) = Y_i'(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}),$$

$$\mu_\rho(x, y_1, \cdots, y_n) = \Lambda_\rho(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}_n).$$

Under these circumstances, if $\mathfrak{T}_0$ *is the transversal surface of the field passing through the end point* P_0 *of the extremal* $\mathfrak{E}_0$, P_2 *any point of* $\mathfrak{T}_0$ *and*

$$\tilde{\mathfrak{E}}: \qquad y_i = \tilde{y}_i(x), \qquad x_2 \leqq x \leqq x_1,$$

any curve of class C', *passing through the point* P_2 *and the second end-point* P_1 *of* $\mathfrak{E}_0$, *which lies wholly in the field* $\mathbf{S}_k$ *and satisfies the partial differential equations*

$$f_\rho\,[x, \tilde{y}_1(x), \cdots;\ \tilde{y}_1'(x), \cdots] = 0,$$

the total variation

$$\Delta J = J_{\tilde{\mathfrak{E}}}(P_2 P_1) - J_{\mathfrak{E}_0}(P_0 P_1)$$

is expressible by the definite integral

$$\Delta J = \int_{x_2}^{x_1} E\{x, \tilde{y}_1(x), \cdots;\ p_1[x, \tilde{y}_1(x), \cdots], \cdots;\ \tilde{y}_1'(x), \cdots;\ \mu_1[x, \tilde{y}(x), \cdots], \cdots\}\,dx,$$

where the E-function is defined by the equation

$$E(x, y_1, \cdots;\ p_1, \cdots;\ \tilde{y}_1', \cdots;\ \mu_1, \cdots) = f(x, y_1, \cdots;\ \tilde{y}_1', \cdots)$$
$$-f(x, y_1, \cdots;\ p_1, \cdots) - \sum_i (y_i' - p_i)\,\Omega_{n+i}(x, y_1, \cdots;\ p_1, \cdots;\ \mu_1, \cdots).$$

From the fundamental formulas (78) follows also immediately the extension of *Beltrami-Hilbert's independence theorem* as given by MAYER:[*]

If p_i, μ_ρ *are the functions of* $x, y_1, \cdots, y_n$ *defined by* (79), *then the expression*

$$T \equiv f(x, y_1, \cdots;\ p_1, \cdots) + \sum_i \left(\frac{dy_i}{dx} - p_i\right)\Omega_{n+i}(x, y_1, \cdots;\ p_1, \cdots;\ \mu_1, \cdots)$$

is a complete derivative with respect to x, *for arbitrary functions* $y_1, \cdots, y_n$ *of* x, *and at the same time*

$$f_\rho(x, y_1, \cdots;\ p_1, \cdots) = 0,$$

identically in $x, y_1, \cdots, y_n$.

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[*] Compare MAYER II, § 3.

For, according to (78)

$$T = \frac{d}{dx} W(x, y_1, \cdots, y_n),$$

whatever functions of x may be substituted for $y_1, \cdots, y_n$, and (82) is obtained from

$$f_\rho(x, Y_1, \cdots; Y'_1, \cdots) = 0$$

by substituting $\mathfrak{a}_i$ for a_i.

Hence it follows that the value of HILBERT's invariant integral generalized,* i. e., the integral

$$J^*_{\mathfrak{C}} = \int_{\mathfrak{C}} T dx,$$

taken along any admissible curve $\mathfrak{C}$ lying wholly in the field S_k, is independent of the path of integration and depends only on the position of the end-points, since

$$J^*_{\mathfrak{C}} = W(a'', b''_1, \cdots, b''_n) - W(a', b'_1, \cdots, b'_n),$$

if $a', b'_1, \cdots, b'_n$ and $a'', b''_1, \cdots, b''_n$ respectively are the coördinates of the first and second end-point of $\mathfrak{C}$.

On applying this result first to the curve $\tilde{\mathfrak{C}}$ and then to the curve $\mathfrak{C}_0$, we obtain

$$J^*_{\tilde{\mathfrak{C}}} = W(x_1, y_{11}, \cdots, y_{n1}) - W(x_2, y_{12}, \cdots, y_{n2}),$$

$$J^*_{\mathfrak{C}_0} = W(x_1, y_{11}, \cdots, y_{n1}) - W(x_0, y_{10}, \cdots, y_{n0}).$$

But since the points P_2 and P_0 lie on the same transversal surface $\mathfrak{T}_0$, we have according to § 7

$$W(x_2, y_{12}, \cdots, y_{n2}) = W(x_0, y_{10}, \cdots, y_{n0}),$$

and therefore

$$J^*_{\mathfrak{C}_0} = J^*_{\tilde{\mathfrak{C}}}.$$

On the other hand it follows from (79) that

$$p_i(x, Y_1, \cdots, Y_n) = Y'$$

and therefore in particular for $a_i = a_i^0$,

$$p_i[x, y_1(x), \cdots, y_n(x)] = y'_i(x).$$

Hence

$$J_{\mathfrak{C}_0} = J^*_{\mathfrak{C}_0} = J^*_{\tilde{\mathfrak{C}}},$$

and therefore

$$\Delta J = J_{\tilde{\mathfrak{C}}} - J^*_{\tilde{\mathfrak{C}}},$$

which is again WEIERSTRASS' theorem in the form given above.

FREIBURG I. B.,
April 27, 1906.

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* Compare MAYER II, p. 66.